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On Linnik's continuous-time random walks

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Abstract. In many fields of applied physics, the phenomenology of the space–time phenomena to be understood (in general for prediction purposes) may be described in the following most simple way: events with random common positive amplitude occur randomly in time according to a continuous time random walk (CTRW) model; the prerequisite is therefore a statistical model for both the amplitude and inter-arrival times between events, here assumed mutually independent. Special attention is paid here to CTRW for which both amplitude and holding time have infinite mean value (the extreme and rare hypothesis). Such processes and their limiting version arise in particular as inverses of processes with stationary independent increments of special interest (chiefly related to the Lévy stable subordinator).

Among other related models, we investigate here some properties of this CTRW in situations where the occurrence of events is modelled by a discrete inverse-Linnik process which shares the rare event hypothesis; this class derives (statistically) its importance from its close relationship to many other meaningful processes such as the Lévy, gamma and Mittag–Leffler ones. Physically, Linnik and inverse-Linnik processes appear as a recurrent paradigm in relaxation theory of condensed matter. The limit laws for cumulative Linnik sequences and their time to failure are finally discussed.

1. Introduction

There are many fields of applied physics where the problem is the statistical understanding of natural phenomena presenting to us as a sequence in time of inherently extremely irregular data in space: in hydrology, this could be the sequence of water inputs into some river or dam, in geophysics, it could be the sequence of random releases of earthquake energy. It could also be the sequence of damages met by the customers of some insurance company in finance, or the random users' demands for network or energy resources in telecommunications' or power supply management technology.

In this paper, we are interested by a continuous time random walk (CTRW) model for which such physical phenomena are to be described in the following realistic way: events of random independent and identically distributed (IID) positive magnitude, occur at random times, the inter-arrival times of which form an IID sequence: for such sequences, both instants of occurrence and amplitudes of the events are under concern. Recently there has been special emphasis in the literature on the possibility that *both* magnitude and inter-arrival times' sequences are heavy-tailed with tail exponents $\alpha > 0$ and $\delta > 0$ respectively, possibly smaller than one (the extreme and rare event hypothesis). Such processes endeavour special statistical properties [16] and are certainly ubiquitous in nature: to take an image borrowed from climatology, storms in arid regions are reputed to occur rarely but with extreme violence. Reports on heavy-tailed amplitudes with exponent $\alpha = \frac{2}{3} < 1$ also exist in the context of earthquake magnitude data [14, 15, 19, 23], in accordance with the Gutenberg–Richter theory [12]. The inter-arrival time problem in this context is also investigated in [16, 30].

In this paper, we focus on a particular CTRW model, namely the inverse-Linnik model for the occurrence of events in time. Our motivations are twofold:

(1) Statistical, as Linnik processes will be shown to embody a large class of statistical phenomena of interest.
(2) Physical, as Linnik distributions and related ones (such as Mittag–Leffler's, Lévy's and gamma's) appear as a paradigm in the context of non-Debye relaxation theory in disordered complex systems (see [20, 21, 35] and the references therein) and in the problems of anomalous diffusion [17, 21]. Our chief goal is to supply some statistical insight into these distributions; these (and other) distributions encountered in this paper are all shown to be intimately twined with scaling Lévy variables.

More precisely, this paper is organized as follows.

In section 2, we recall some known properties of the CTRW model: the basic object which embodies all statistical informations on CTRW is their Laplace functional which may be computed from local statistical informations on their initial condition, holding time and jump-height distributions.

In section 3, we focus on the central notion of the inverse process of an original process. Special attention is paid here to situations where the original process to be inverted has stationary independent increments (SII) and non-decreasing sample paths, i.e. is a subordinator. It turns out that the inverse of such subordinators with SII is also a subordinator in the CTRW class, reversing the roles played by space and time. To illustrate this point, we first compute the Laplace functional of the inverse when the original process is a simple compound Poisson subordinator. When the original process is a Lévy stable subordinator, it may be interpreted as a limiting compound Poisson process; consequently, the inverse-Lévy subordinator is also a limiting CTRW whose construction is supplied. Similarly, gamma and inverse-gamma subordinators can be studied. Finally, composing two subordinators yields a new subordinator; this observation is exploited to study the CTRW obtained when composing a Lévy with an inverse-Lévy process. These constructions will prove useful in what follows as they give some statistical insight into *mixed* inverse-Lévy distributions.

As we wish to pay special attention to Linnik processes, the whole of section 4 is devoted to them. It turns out that the statistical properties of the inverse process are similarly understandable when the original process is a Linnik subordinator with SII, obtained when composing Lévy and gamma subordinators. Linnik processes are infinitely divisible (ID) (and even self-decomposable (SD)); they include as particular cases both Mittag–Leffler and Lévy stable processes. They are strongly connected with the Cole–Cole relaxation function. In some limiting sense, Linnik subordinators exhibit heavy-tailed amplitude jumps (the extreme events situation). Conversely, inverse-Linnik subordinators exhibit heavy-tailed holding times between consecutive jumps (the rare event situation).

A discrete inverse-Linnik process is then introduced. It is obtained when composing a standard Poisson process with an inverse-Linnik subordinator, independently. It gives an interesting counting Linnik model relating to the way events occur randomly and rarely in time.

In section 5, we address the following problem: assume events occur randomly in time according to a discrete inverse-Linnik model. Suppose some statistical (possibly heavy-tailed) model for the positive magnitude of such events holds. The cumulative partial sums of such Linnik sequences are then of physical interest. We thus derive the various asymptotics to be expected were the magnitude partial sums to be observed from the origin of time and over a large period of time $t > 0$ and exhibit the various time-scaling properties of these variables: whereas the heavy-tailed magnitude hypothesis tends to accelerate the partial sums growth with time, heavy-tailed holding times tend to slow this process, with a competition between the two. Here, the limit distributions of the (rescaled) partial sums are properly and easily identified as inverse-Lévy and *mixed* inverse-Lévy distributions as mentioned at the end of section 3.

In section 6, we also give an answer to the 'time to failure question for inverse-Linnik sequences' which is important in practice: how long should one wait before the first event of magnitude exceeding some (possibly high) threshold is achieved; it turns out that the exact distribution is available; the asymptotic behaviour of this variable, properly rescaled, is also identified as a Mittag–Leffler variable. These constitute important issues in the prediction of floods, catastrophic earthquakes, ruin or buffer overflows in the examples just mentioned, were the Linnik model to hold.

2. Renewal processes (CTRW) revisited

We first recall salient facts arising from the modelling of events occurring randomly in time [6, 10, 31, 32].

2.1. Counting events

Suppose at time $t = 0$, some event occurs for the first time. Suppose successive events occur in the future in such a way that the inter-arrival times between consecutive events form an IID sequence $(\boldsymbol{\tau}_m, m \geqslant 1)$ with common distribution $\boldsymbol{\tau}_m \stackrel{d}{=} \boldsymbol{\tau}, m \geqslant 1$. The inter-arrival time $\boldsymbol{\tau}$ is assumed to have a density function, say $f_{\boldsymbol{\tau}}(t)$.

We are then left with a sequence of events occurring at times

$$T_0 = 0 \qquad T_n := \sum_{m=1}^{n} \boldsymbol{\tau}_m \qquad n \geqslant 1. \tag{1}$$

Let $N(t)$, $t \geqslant 0$, count the random number of events which occurred in the time interval $[0, t]$. Clearly,

$$N(t) = \sum_{n \geqslant 0} \mathbf{1}(T_n \leqslant t) \tag{2}$$

with $\mathbf{1}(.)$ the set indicator function which takes the value one if the event is realized, zero, otherwise.

As a result, an essential feature of such processes is that the events '$N(t) > n$' and '$T_n \leqslant t$' coincide.

Such random processes are called *pure* counting renewal processes (the adjective pure is relative to the hypothesis which has been made that the origin of time is an instant at which some event occurred; if this not the case, the adjective *delayed* is currently employed and the first event occurs at time $T_0 := \boldsymbol{\tau}_0 > 0$, independent of $(\boldsymbol{\tau}_m, m \geqslant 1)$ but not necessarily with the same distribution). If in addition $\int_0^{+\infty} f_{\boldsymbol{\tau}}(s)\,\mathrm{d}s = 1$ ($\boldsymbol{\tau}$ is 'proper') such renewal processes are said to be recurrent; this has to be opposed to transient renewal processes for which $\int_0^{+\infty} f_{\boldsymbol{\tau}}(t)\,\mathrm{d}t < 1$, corresponding to 'defective' $\boldsymbol{\tau}$, allowing for a finite probability that the first event never occurs, i.e. occurs at time $t = +\infty$. We shall avoid delayed and transient processes in what follows and limit ourselves to pure and recurrent ones. However, among recurrent processes, we shall distinguish between positive recurrent processes for which the average renewal time $\langle \boldsymbol{\tau} \rangle := \theta < +\infty$ and null recurrent for which $\langle \boldsymbol{\tau} \rangle = +\infty$.

If $\langle \boldsymbol{\tau} \rangle = +\infty$, we shall limit ourselves to situations where this occurs as a result of 'heavy-tailedness' of the inter-arrival time with tail distribution $\Pr(\boldsymbol{\tau} > t) \sim c_\delta t^{-\delta}$, as $t \uparrow +\infty$, with $\delta \in (0, 1)$. Here, $c_\delta > 0$ is a scale factor for $\boldsymbol{\tau}$. In other words $c_\delta = t_0^\delta$ for some $t_0 > 0$ fixing the timescale itself.

We note that if the inter-arrival times $(\boldsymbol{\tau}_m, m \geqslant 1)$ are exponentially distributed, this counting process boils down to the familiar Poisson process.

We shall call $\Lambda(t) := \langle N(t) \rangle$, $t \geqslant 0$, the intensity of the pure renewal process and $\lambda(t) := \mathrm{d}\Lambda(t)/\mathrm{d}t$ its rate (i.e. the instantaneous 'frequency' at which events occur at time t); the function $\Lambda(t)/t$ is called the frequency of the phenomenon.

It follows from (2) that

$$\Lambda(t) = \mathbf{1}(t \geqslant 0) + \sum_{n \geqslant 1} \int_0^t f_{\boldsymbol{\tau}}^{*n}(s)\,\mathrm{d}s \tag{3}$$

$$\lambda(t) = \delta(t = 0) + \sum_{n \geqslant 1} f_{\boldsymbol{\tau}}^{*n}(s)\,\mathrm{d}s \tag{4}$$

where $f_{\boldsymbol{\tau}}^{*n}$ is the n-fold convolution of $f_{\boldsymbol{\tau}}$ (i.e. is the density of T_n), $\mathbf{1}(t \geqslant 0)$ the Heavyside step function, $\delta(t = 0)$ the Dirac delta function at $t = 0$.

If we let $\hat{\Lambda}(p) := \int_0^{+\infty} \mathrm{e}^{-pt}\Lambda(t)\,\mathrm{d}t$, $\hat{\lambda}(p) := \int_0^{+\infty} \mathrm{e}^{-pt}\lambda(t)\,\mathrm{d}t$ and $\hat{f}_{\boldsymbol{\tau}}(p) := \int_0^{+\infty} \mathrm{e}^{-pt} f_{\boldsymbol{\tau}}(t)\,\mathrm{d}t$ stand for the Laplace transforms of $\Lambda(t)$, $\lambda(t)$ and $f_{\boldsymbol{\tau}}(t)$, respectively, the last two equations yield

$$\hat{\Lambda}(p) = \frac{1}{p(1 - \hat{f}_{\boldsymbol{\tau}}(p))} \qquad \text{and} \qquad \hat{\lambda}(p) = \frac{1}{1 - \hat{f}_{\boldsymbol{\tau}}(p)} \tag{5}$$

underlining the connection between probability theory and physical rate processes. This suggests the following remark: in many applications the distribution of the renewal time $\boldsymbol{\tau}$ is unknown or too complicated to determine or imagine; however, the intensity function $\Lambda(t)$ may sometimes be easily obtained from the data. From (5), the Laplace time transform of $\boldsymbol{\tau}$ may be conversely computed from the one $\hat{\Lambda}(p)$ of the intensity and is $\hat{f}_{\boldsymbol{\tau}}(p) = 1 - 1/(p\hat{\Lambda}(p))$.

As p tends to zero, we get some information on the way the intensity and rate functions behave for large times [10]. These are strongly connected to the tail distribution of the variable $\boldsymbol{\tau}$ and we have to distinguish between two cases.

(1) $\theta < +\infty$. In this case, $\hat{f}_{\boldsymbol{\tau}}(p) \sim 1 - \theta p$, as $p \downarrow 0$. Hence, $\hat{\Lambda}(p) \sim 1/(\theta p^2)$ and $\hat{\lambda}(p) \sim 1/(\theta p)$, as $p \downarrow 0$. This means $\Lambda(t) \sim t/\theta$ and $\lambda(t) \sim 1/\theta$ as $t \uparrow +\infty$. For recurrent positive processes, the rate function tends to $1/\theta$ as time drifts to infinity.

(2) $\theta = +\infty$. In this case, i.e. for recurrent null processes, the rate function tends to zero: this is a 'rare event' hypothesis, as the expected time between consecutive events is infinite.

For example, if time $\boldsymbol{\tau}$ is such that $\Pr(\boldsymbol{\tau} > t) \sim c_\delta t^{-\delta}$, as $t \uparrow +\infty$, with $\delta \in (0, 1)$, $c_\delta > 0$, in such a way that $\theta = +\infty$, then $\hat{f}_{\boldsymbol{\tau}}(p) \sim 1 - c_\delta p^\delta$, as $p \downarrow 0$. Hence, $\hat{\Lambda}(p) \sim 1/(c_\delta p^{\delta+1})$ and $\hat{\lambda}(p) \sim 1/(c_\delta p^\delta)$, as $p \downarrow 0$. This means $\Lambda(t) \sim t^\delta/c_\delta$ and $\lambda(t) \sim t^{\delta-1}/c_\delta$ as $t \uparrow +\infty$: the intensity goes to infinity slower than t and the rate function tends to zero algebraically. As time goes to infinity, the events get sparser and sparser, owing to the infinite average hypothesis of the inter-arrival times.

2.2. Cumulating magnitude

The process $N(t)$ counts the number of events which occurred before time t: each time an event occurs, the counter is incremented by unity. Assume now some physical phenomenon to be described by a compound renewal process: events of random IID magnitude, say $(\boldsymbol{\chi}_m, m \geqslant 1)$, occur at random times $T_n, n \geqslant 1$, the inter-arrival times of which form an IID sequence. This model was introduced in physics in [27], and its properties are extensively examined in the context of CTRW models, including the rareness hypothesis, [22, 34], in the fractal time random walk (FTRW) model. One may be interested by a process which cumulates this random number of random amplitudes. A compound renewal process is to the counting renewal process what a compound Poisson process is to a Poisson process itself. Physical situations where the relevance of this model holds are numerous: think of the random magnitude as a claims' sequence in insurance risk theory, as the energy release of individual earthquakes in geophysics or as random water inputs flowing into a dam in hydrology. Summing the individual contributions yields the total claim amount (cumulative energy release and water input) over a certain laps of time. In all these applications we have in mind, the magnitude χ is a *positive* random variable; we shall therefore only deal with this case in what follows. As a result the Laplace–Stieltjes transform of the magnitude distributions will be employed rather than the Fourier transform which is adapted to real-valued amplitudes sequences not dealt with here. We shall assume that the random local magnitude χ admits a density function, say $f_\chi(x)$.

Partitioning over the first steps of the walk, the cumulative magnitude $X(t)$ may be defined in distribution by

$$X(t) \stackrel{d}{=} \chi_0 \cdot \mathbf{1}(\tau > t) + (\chi(\tau) + X(t-\tau)) \cdot \mathbf{1}(\tau \leqslant t) \tag{6}$$

where $\tau > 0$ is a 'proper' positive random variable known as the first renewal time of $X(t)$. Such processes are called compound pure recurrent renewal processes.

Let us briefly comment this identity distribution. At time τ, $X(t)$ undergoes a first (random) jump with amplitude $\chi(\tau) > 0$, possibly dependent on the occurrence time τ of this jump. Now, fix time t at which $X(t)$ is to be evaluated. If the realization of time τ exceeds the time t of interest, the process $X(t)$ is in the initial state, say χ_0. If $\tau = s \leqslant t$, the value of $X(t)$ is the independent sum of the first jump of amplitude $\chi(s)$ plus a statistical copy of the process $X(.)$ in the remaining time $t-s$, conditionally to the event $\tau = s$. This is a reasonable way to see a renewal process, as a statistical copy of itself after the first renewal time; it generalizes the familiar compound Poisson process family in that the inter-arrival time distributions between spikes is an IID sequence but not necessarily exponentially distributed. From this model, it is clear that $X(t) = \chi_0 + \sum_{m=1}^{N(t)} \chi_m$, where $(\chi_m, m \geqslant 1)$ is the IID random magnitude' sequence.

Let us now translate the definition (6) in terms of the evolution of the Laplace transform of $X(t)$. Let

$$\Phi_X(t,\lambda) := \langle \mathrm{e}^{-\lambda X(t)} \rangle \qquad \text{and} \qquad \hat{f}_\chi(s,\lambda) := \langle \mathrm{e}^{-\lambda \chi(s)} \rangle \tag{7}$$

respectively stand for the Laplace transforms of the cumulative process $X(t)$ and of a local magnitude $\chi(s)$ which occurred at time $s \leqslant t$.

Upon conditioning with respect to the various possible realizations of τ, equation (6) yields an integral evolution equation for the density of $X(t)$, which, in terms of $\langle \mathrm{e}^{-\lambda X(t)} \rangle$, reads

$$\Phi_X(t,\lambda) = \hat{f}_{\chi_0}(\lambda)\,\mathrm{Pr}(\tau > t) + \int_0^t \Phi_X(t-s,\lambda)\hat{f}_\chi(s,\lambda) f_\tau(s)\,\mathrm{d}s \tag{8}$$

where $\hat{f}_{\chi_0}(\lambda) := \langle \exp -\lambda\chi_0 \rangle$.

We shall now make an additional simplifying hypothesis.

Assume that the local magnitude are independent of their occurrence time (the *decoupling* hypothesis in the CTRW model); then $\hat{f}_\chi(s,\lambda) = \hat{f}_\chi(\lambda)$ and the Laplace transform of the conditional magnitude χ is independent of the particular realization s of its occurrence time τ. Then (8) reduces to

$$\Phi_X(t,\lambda) = \hat{f}_{\chi_0}(\lambda)\Pr(\tau > t) + \hat{f}_\chi(\lambda)\int_0^t \Phi_X(t-s,\lambda) f_\tau(s)\,\mathrm{d}s. \tag{9}$$

This is the simpler convolution integral equation which $\Phi_X(t,\lambda)$ now satisfies. Introducing the Laplace transforms

$$\hat{\Phi}_X(p,\lambda) := \int_0^{+\infty} \mathrm{e}^{-pt}\Phi_X(t,\lambda)\,\mathrm{d}t \qquad \text{and} \qquad \hat{f}_\tau(p) := \langle \mathrm{e}^{-p\tau}\rangle \tag{10}$$

respectively of $\Phi_X(.,\lambda)$ and $f_\tau(.)$, (9) yields

$$\hat{\Phi}_X(p,\lambda) = \frac{(1-\hat{f}_\tau(p))\hat{f}_{\chi_0}(\lambda)}{p(1-\hat{f}_\tau(p)\hat{f}_\chi(\lambda))} \tag{11}$$

provided $\hat{f}_\tau(p)\hat{f}_\chi(\lambda) < 1$.

We shall call $\hat{\Phi}_X(p,\lambda)$ the Laplace functional of the process $X(t)$, $t \geqslant 0$. In addition, the triptych $(\hat{f}_{\chi_0}(\lambda), \hat{f}_\chi(\lambda), \hat{f}_\tau(p))$ will be called the space–time characteristics of the process $X(t)$, $t \geqslant 0$ as they completely determine its law.

Processes whose Laplace functional is given by (11) are known as pure *renewal* processes with stationary local magnitude [11], or CTRW. This expression is consistent with similar results in the lattice case to be found in [13].

Remark 1. Renewal process are more general than standard processes with SII, such as Poisson, because they are not Markovian as the integral equation (8) shows: the distribution at time t of $X(t)$ depends (in general) on the distribution of $X(s)$, $s < t$. However, it can easily be shown that they include the compound Poisson class (a very important sub-class of processes with SII) which may be recovered if the renewal time τ is assumed to be exponentially distributed, because of the memory-less character of the exponential distribution. If T is exponentially distributed, then $\hat{f}_\tau(p) = 1/(1+\theta p)$ and (11) boils down to

$$\hat{\Phi}_X(p,\lambda) = \frac{\theta}{p\theta + 1 - \hat{f}_\chi(\lambda)} \tag{12}$$

provided $\chi_0 = 0$.

As a result, the Laplace transform of $X(t)$ is the one of a compound Poisson process

$$\langle \mathrm{e}^{-\lambda X(t)}\rangle = \exp -\frac{t}{\theta}(1-\hat{f}_\chi(\lambda)). \tag{13}$$

The standard integer-valued Poisson process, say $P(t)$, is obtained while assuming the degenerate form $\chi := \eta = 1$ of the jump amplitude in this last expression:

$$\langle \mathrm{e}^{-\lambda P(t)}\rangle := \exp -\frac{t}{\theta}(1-\mathrm{e}^{-\lambda}). \tag{14}$$

2.3. Compound counting

Note that the pure elementary counting renewal process may be recovered from $\chi_0 = 1$ and $\chi = 1$ which states that the initial condition of the counting process is one, with a degenerate magnitude $\chi = \eta = 1$ which takes the value one when an event occurs. We shall call $\Phi_N(t,\lambda) := \langle \mathrm{e}^{-\lambda N(t)}\rangle$, $\lambda \geqslant 0$, so that in this case (11) reads

$$\hat{\Phi}_N(p,\lambda) = \frac{(1-\hat{f}_\tau(p))\mathrm{e}^{-\lambda}}{p(1-\hat{f}_\tau(p)\mathrm{e}^{-\lambda})} \tag{15}$$

provided $\hat{f}_\tau(p)\mathrm{e}^{-\lambda} < 1$. Thus the equation $\hat{f}_\tau(p) = \exp\lambda$ is the location of the poles of $\hat{\Phi}_N(p,\lambda)$.

If the increment χ (initial condition χ_0) now takes random non-degenerate integral values, we shall call it η (respectively η_0). If this is so, the cumulative process $X(t)$ takes itself integral values (and we shall therefore call it $\mathfrak{N}(t)$ to avoid confusion); it is then a compound recurrent *counting* renewal process: in this model, at renewal times T_n, $n \geqslant 1$, it is assumed that a random number $\eta \geqslant \mathbf{1}$ of events may occur simultaneously. With $\Phi_{\mathfrak{N}}(t,\lambda) := \langle \mathrm{e}^{-\lambda\mathfrak{N}(t)}\rangle$, $\lambda \geqslant 0$, we then get

$$\hat{\Phi}_{\mathfrak{N}}(p,\lambda) = \frac{(1-\hat{f}_\tau(p))\hat{f}_{\eta_0}(\lambda)}{p(1-\hat{f}_\tau(p)\hat{f}_\eta(\lambda))} \tag{16}$$

where $\hat{f}_\eta(\lambda) := \langle \mathrm{e}^{-\lambda\eta}\rangle$ is the Laplace transform of the discrete jumps' distribution. The triptych $(\hat{f}_{\eta_0}(\lambda), \hat{f}_\eta(\lambda), \hat{f}_\tau(p))$ will be called the space–time characteristics of the process $\mathfrak{N}(t)$, $t \geqslant 0$ as they completely determine its law.

2.4. Cumulative magnitude process driven by a compound counting process

Let $\mathfrak{N}(t)$ be a compound counting renewal process, whose characteristics are given by $(\hat{f}_{\eta_0}(\lambda) = \hat{f}_\eta(\lambda), \hat{f}_\tau(p))$. Let $(\chi_m, m \geqslant 0)$ be an IID sequence. The process

$$\mathfrak{X}(t) = \sum_{m=0}^{\mathfrak{N}(t)} \chi_m \tag{17}$$

is called a cumulative magnitude process driven by $\mathfrak{N}(t)$.

We then have the result of the following proposition.

Proposition 1. *Let $\hat{f}_\chi(\lambda) := \langle \mathrm{e}^{-\lambda\chi}\rangle$. Let $\hat{f}_{\mathfrak{X}}(\lambda) = \hat{f}_\eta(-\log \hat{f}_\chi(\lambda))$ be the Laplace transform of the random sum*

$$\mathfrak{X} = \sum_{m=0}^{\eta} \chi_m. \tag{18}$$

Then, the process $\mathfrak{X}(t)$ is a compound renewal process with space–time characteristics

$$\hat{f}_{\mathfrak{X}_0}(\lambda) = \hat{f}_{\mathfrak{X}}(\lambda) = \hat{f}_\eta(-\log \hat{f}_\chi(\lambda)) \qquad \hat{f}_\tau(p) \tag{19}$$

in such a way that $\mathfrak{X}(t) = \sum_{m=0}^{N(t)} \mathfrak{X}_m$ and $\mathfrak{N}(t) = \sum_{m=0}^{N(t)} \eta_m$.

Proof. We have

$$\langle \mathrm{e}^{-\lambda\mathfrak{X}(t)}\rangle = \sum_{n\geqslant 0} \hat{f}_\chi(\lambda)^n \Pr(\mathfrak{N}(t) = n) = \Phi_{\mathfrak{N}}(t, -\log \hat{f}_\chi(\lambda)). \tag{20}$$

Thus

$$\hat{\Phi}_{\mathfrak{X}}(p,\lambda) = \hat{\Phi}_{\mathfrak{N}}(p, -\log \hat{f}_\chi(\lambda)). \tag{21}$$

From (11) and (16), the result follows. □

From this result, it follows that a cumulative magnitude process driven by a compound counting renewal process boils down to a standard cumulative magnitude process driven by the elementary counting renewal process, with amplitude properly readjusted along (18), which is fair enough.

3. Inverse processes

In this section, we focus on the notion of the inverse of a process which is a central one for our purpose; special attention is paid to situations where the original process to be inverted has stationary independent increments, and non-decreasing sample paths, i.e. is a subordinator (see [1,26, p 366]). We supply a few examples of interest in preparation for the considerations on Linnik and inverse-Linnik subordinators to be made in the next section.

3.1. Generalities on inverse processes

Let $X(t)$, $t \geqslant 0$ be a real-valued stochastic process, such that $X(0) = 0$. With $x \geqslant 0$, let

$$T(x) := \inf(t > 0 : X(t) > x). \tag{22}$$

This process has non-decreasing sample paths and so is said to be a subordinator. Actually, $T(x)$ is the generalized inverse of the extremal process

$$X^*(t) := \max_{s \leqslant t} X(s) \tag{23}$$

which has also non-decreasing sample paths. Note that if $X(t)$ is itself non-decreasing sample-paths, then the two processes $X(t)$ and $X^*(t)$ share the same sample paths. In this case, $T(x)$ is directly the inverse of $X(t)$.

The result of the following proposition, relating the Laplace functional of $X^*(t)$ and $T(x)$, holds.

Proposition 2. *Let $X(t)$, $X^*(t)$ and $T(x)$ be defined as above. Let $\hat{\Phi}_{X^*}(p, \lambda)$ stand for the Laplace transform in time of $\Phi_{X^*}(t, \lambda) := \langle e^{-\lambda X^*(t)} \rangle$ and $\hat{\Phi}_T(\lambda, p)$ stand for the Laplace transform in space of $\Phi_T(x, p) := \langle e^{-pT(x)} \rangle$. Then, the formula*

$$\lambda \hat{\Phi}_T(\lambda, p) + p\hat{\Phi}_{X^*}(p, \lambda) = 1 \tag{24}$$

holds.

Proof. To see this, we observe that the events '$T(x) > t$' and '$X^*(t) \leqslant x$' coincide; thus, we get the identity

$$\int_0^{+\infty} e^{-\lambda x} \Pr(T(x) > t) \, dx = \int_0^{+\infty} e^{-\lambda x} \Pr(X^*(t) \leqslant x) \, dx = \frac{1}{\lambda} \langle e^{-\lambda X^*(t)} \rangle. \tag{25}$$

Upon taking the Laplace transform in time, and permuting the integrals

$$\int_0^{+\infty} e^{-pt} \, dt \int_0^{+\infty} e^{-\lambda x} \Pr(T(x) > t) \, dx = \int_0^{+\infty} e^{-\lambda x} \, dx \frac{1}{p} [1 - \langle e^{-pT(x)} \rangle] \tag{26}$$

$$= \frac{1}{p\lambda} - \frac{1}{p} \hat{\Phi}_T(\lambda, p) = \frac{1}{\lambda} \hat{\Phi}_{X^*}(p, \lambda). \tag{27}$$

We thus prove the assertion. □

Let $X(t)$ be a real-valued stochastic process, such that $X(0) = 0$. Compute the Laplace functional of $T(x)$. From (24), this is

$$\hat{\Phi}_T(\lambda, p) = \frac{1}{\lambda}(1 - p\hat{\Phi}_{X^*}(p, \lambda)). \tag{28}$$

Upon exchanging the roles of λ and p (that is to say space and time), we get the Laplace functional of a standard process, say $Z(t)$, in the temporal domain

$$\hat{\Phi}_Z(p, \lambda) := \frac{1}{p}(1 - \lambda \hat{\Phi}_{X^*}(\lambda, p)). \tag{29}$$

It is such that

$$\hat{\Phi}_Z(p,\lambda) = \int_0^{+\infty} \mathrm{e}^{-pt} \langle \mathrm{e}^{-\lambda Z(t)} \rangle \, \mathrm{d}t \tag{30}$$

and the process $Z(t)$, $t \geqslant 0$ is said to be the inverse of the process $X(t)$, $t \geqslant 0$.

3.2. Fundamental examples

In this section, we supply some simple and fundamental examples, starting with the simplest one.

Example 1. Compound Poisson and inverse Poisson subordinators.

Assume $X(t)$ is a compound Poisson subordinator hence with stationary independent positive increments and non-decreasing sample-paths; then, from (13),

$$\langle \mathrm{e}^{-\lambda X(t)} \rangle = \langle \mathrm{e}^{-\lambda X^*(t)} \rangle = \mathrm{e}^{-\frac{t}{\theta}(1-\hat{f}_\chi(\lambda))} \tag{31}$$

with $\hat{f}_\chi(\lambda)$ the Laplace–Stieltjes transform of a positive increment random variable (the jump height), say χ. As a result, $X^*(.) = X(.)$, and

$$\hat{\Phi}_{X^*}(p,\lambda) = \frac{\theta}{p\theta + 1 - \hat{f}_\chi(\lambda)}. \tag{32}$$

The Laplace functional of the inverse process easily follows from (29). It can be expressed as

$$\hat{\Phi}_Z(p,\lambda) = \frac{1}{p}\left(1 - \lambda \frac{\theta}{\lambda\theta + 1 - \hat{f}_\chi(p)}\right) = \frac{1 - \hat{f}_\chi(p)}{p(\lambda\theta + 1 - \hat{f}_\chi(p))}. \tag{33}$$

With $\hat{f}(\lambda) = 1/(1+\lambda\theta)$, the Laplace–Stieltjes transform of the exponential distribution, with mean value θ, this is also

$$\hat{\Phi}_Z(p,\lambda) = \frac{(1 - \hat{f}_\chi(p))\hat{f}(\lambda)}{p(1 - \hat{f}_\chi(p)\hat{f}(\lambda))}. \tag{34}$$

Hence, from (11), it is the Laplace functional of a compound renewal process such that

$$\hat{\Phi}_Z(p,\lambda) = \frac{(1 - \hat{f}_\tau(p))\hat{f}_{\zeta_0}(\lambda)}{p(1 - \hat{f}_\tau(p)\hat{f}\zeta(\lambda))} \tag{35}$$

with space–time characteristics

$$\hat{f}_{\zeta_0}(\lambda) = \hat{f}_\zeta(\lambda) = \frac{1}{(1+\lambda\theta)} \qquad \text{and} \qquad \hat{f}_\tau(p) = \hat{f}_\chi(p) \tag{36}$$

simply exchanging the roles of space and time.

The inverse of a compound Poisson subordinator is a renewal process which exhibits initial and subsequent IID exponentially distributed jump-heights ζ_0, ζ. The holding time τ between consecutive jumps has the distribution of the jump-height χ of the original compound Poisson subordinator.

We now come to a more intricate example.

Example 2. Lévy-stable and inverse-Lévy-stable subordinators.

Assume $X(t)$ is now a standard Lévy-stable subordinator. Then, with $\delta \in (0, 1)$

$$\langle \mathrm{e}^{-\lambda X(t)} \rangle = \langle \mathrm{e}^{-\lambda X^*(t)} \rangle = \mathrm{e}^{-t\lambda^\delta}. \tag{37}$$

Note from (37) that

$$X(t) \stackrel{d}{=} t^{1/\delta} L_\delta \qquad t \geqslant 0 \tag{38}$$

where L_δ is a standard positive Lévy variable, with

$$\langle e^{-\lambda L_\delta} \rangle = \exp -\lambda^\delta. \tag{39}$$

Remark 2. Before discussing the particular status of the inverse-Lévy subordinator, it should first be emphasized that a Lévy subordinator is a limiting compound Poisson process [10, 29].

First recall the identity (which may be checked upon deriving with respect to λ)

$$\int_0^{+\infty} (1 - e^{-\lambda x}) \rho_\chi(x)\, dx = \lambda^\delta \qquad \text{with} \quad \rho_\chi(x) = \frac{\delta}{\Gamma(1-\delta)} x^{-(1+\delta)}.$$

Here $\rho_\chi(x)\, dx$ is a positive Radon exponent measure on $(0, +\infty)$, with infinite total mass, due to the algebraic divergence of its density in the vicinity of zero. As ρ_χ is not a probability density, a Lévy subordinator, for which (37) holds, is not stricto sensu a compound Poisson process (compare with (31)); rather, it can be obtained from a 'coarse' compound Poisson process $X^\varepsilon(t)$ in the limit $\varepsilon \downarrow 0$.

Let indeed $\varepsilon > 0$; consider the compound Poisson process $X^\varepsilon(t)$, $t \geqslant 0$ defined by an exponentially distributed holding time with mean θ^ε and with IID positive increments, say χ^ε, with normalized *truncated* probability density

$$f_{\chi^\varepsilon}(x) = \theta^\varepsilon \cdot \rho_\chi(x) \cdot \mathbf{1}(x > \varepsilon). \tag{40}$$

We note that this is a Pareto distribution in the heavy-tailed class with tail index δ: jumps with very large amplitude are very likely to occur in the sample paths of $X^\varepsilon(.)$.

From (40) the normalization constant is easily obtained. It is

$$\theta^\varepsilon = 1 \Big/ \int_\varepsilon^{+\infty} \rho_\chi(x)\, dx = \Gamma(1-\delta)\varepsilon^\delta$$

and θ^ε tends to zero as ε tends to zero. Next, we form the quantity $\langle e^{-\lambda X^\varepsilon(t)} \rangle$. From (31), with $\hat{f}_{\chi^\varepsilon}(\lambda) := \langle e^{-\lambda \chi^\varepsilon} \rangle$, it is the Laplace transform of the density f_{χ^ε}

$$\langle e^{-\lambda X^\varepsilon(t)} \rangle = e^{-t/\theta^\varepsilon (1 - \hat{f}_{\chi^\varepsilon}(\lambda))}.$$

Now,

$$1/\theta^\varepsilon (1 - \hat{f}_{\chi^\varepsilon}(\lambda)) = \int_\varepsilon^{+\infty} (1 - e^{-\lambda x}) \rho_\chi(x)\, dx \underset{\varepsilon \downarrow 0}{\to} \lambda^\delta$$

which is consistent with (37).

Thus, $X(t)$ defined from (37) is the limiting compound Poisson process $X^\varepsilon(t)$ as $\varepsilon \downarrow 0$ and is indeed a subordinator with IID increments. As a result, $X(t)$ can be obtained from the renewal process $X^\varepsilon(t)$ with Pareto-exponential space–time characteristics:

$$\hat{f}_{\chi_0}(\lambda) = 1 \qquad \hat{f}_{\chi^\varepsilon}(\lambda) \qquad \text{and} \qquad \hat{f}_\varepsilon(p) = 1/(1 + p\theta^\varepsilon)$$

taking the limit $\varepsilon \downarrow 0$. Compound Poisson subordinators exhibit finitely many isolated jumps on finite-time intervals. This is not the case for Lévy ones: the many jumps with tiny amplitudes contribute in the limit to a Hölder continuous drift, with Hölder exponents in the range $[0, \frac{1}{\delta}]$ [18], as a result of jumps' clustering. This drift is punctuated with a few very large Pareto jumps. Globally, from (38), the process drifts to infinity much faster than clock time t, as a result of the very large jumps which occur.

Let us now come to the inverse asymmetric Lévy subordinator. Its Laplace functional is, from (29) and (37),

$$\hat{\Phi}_Z(p,\lambda) = \frac{1}{p}\left(1 - \lambda\frac{1}{\lambda + p^\delta}\right) = \frac{1}{p(1+\lambda p^{-\delta})}. \tag{41}$$

Hence, as it can easily be checked from (11), example 1 and remark 2, $Z(t)$ as defined from (41), costitute a limiting renewal process $Z^\varepsilon(t)$ as $\varepsilon \downarrow 0$. Here $Z^\varepsilon(t)$ is a renewal process whose coarse space–time characteristics are given by

$$\hat{f}_{\zeta^\varepsilon}(\lambda) = \hat{f}_{\zeta_0^\varepsilon}(\lambda) = \frac{1}{1+\lambda\theta^\varepsilon} \qquad \text{and} \qquad \hat{f}_{\tau^\varepsilon}(p) = \hat{f}_{\chi^\varepsilon}(p) \tag{42}$$

with $\hat{f}_{\chi^\varepsilon}(p) = \langle \mathrm{e}^{-p\chi^\varepsilon}\rangle$ the Laplace transform of f_{χ^ε} defined in (40).

The inverse $Z^\varepsilon(t)$ of the compound Poisson subordinator $X^\varepsilon(t)$ is a renewal process which exhibits initial and subsequent IID exponential jump-heights $\zeta_0^\varepsilon, \zeta^\varepsilon$ with mean θ^ϵ. The waiting time τ^ϵ between its consecutive small exponential jumps has the distribution (40) of the jump amplitude χ^ε of the original compound Poisson subordinator $X^\varepsilon(t)$.

Finally, $Z(t)$ is obtained as the limiting renewal process $Z^\varepsilon(t)$ as $\varepsilon \downarrow 0$. Indeed,

$$\hat{\Phi}_{Z^\varepsilon}(p,\lambda) := \frac{(1-\hat{f}_{\tau^\varepsilon}(p))\hat{f}_{\zeta_0^\varepsilon}(\lambda)}{p(1-\hat{f}_{\tau^\varepsilon}(p)\hat{f}_{\zeta^\varepsilon}(\lambda))} = \frac{1/\theta^\varepsilon(1-\hat{f}_{\chi^\varepsilon}(p))}{p(1/\theta^\varepsilon(1-\hat{f}_{\chi^\varepsilon}(p))+\lambda)} \underset{\varepsilon\downarrow 0}{\to} \frac{1}{p(1+\lambda p^{-\delta})}.$$

Equation (41) characterizes an inverse-Lévy subordinator. Equations (40) and (42) characterize its coarse version $Z^\varepsilon(t), t \geqslant 0$.

Additional insight into the process $Z(.)$ arises from the well known result [10]

$$\langle \mathrm{e}^{-\lambda Z(t)}\rangle = \phi_\delta(\lambda t^\delta) \tag{43}$$

where

$$\phi_\delta(\lambda) := \sum_{n\geqslant 0} \frac{1}{\Gamma(1+n\delta)}(-\lambda)^n \tag{44}$$

is the Mittag–Leffler function (which reduces to the exponential function if $\delta \uparrow 1$). It is also known from [10] that $\phi_\delta(\lambda)$ is the Laplace–Stieltjes transform of an inverse-Lévy variable I_δ, i.e. $\phi_\delta(\lambda) = \langle \mathrm{e}^{-\lambda I_\delta}\rangle$, where $I_\delta := L_\delta^{-\delta}$. Here, L_δ is the standard Lévy variable (39). Thus, from (43)

$$Z(t) \overset{d}{=} t^\delta I_\delta \qquad t \geqslant 0 \tag{45}$$

which should be compared with (38) for the original process. An inverse-Lévy subordinator exhibits a non-decreasing Hölder continuous drift, punctuated with large periods of time (stages) over which it remains constant. Globally, from (45), it drifts to infinity much slower than time t, as a result of the very large stages of its sample paths.

Example 3. Gamma and inverse-gamma processes.

Assume $X(t)$ is a standard ID gamma subordinator. This means, with $\gamma, \theta > 0$

$$\langle \mathrm{e}^{-\lambda X(t)}\rangle = \langle \mathrm{e}^{-\lambda X^*(t)}\rangle = (1+\lambda\theta)^{-\gamma t} = \mathrm{e}^{-\gamma t \log(1+\lambda\theta)}. \tag{46}$$

Recall that $(1+\lambda\theta)^{-\gamma}$ is the Laplace transform of a gamma probability density function, which is known [25] to be ID. Besides $\langle X(t)\rangle = \gamma\theta t$ and this process drifts to infinity like clock time.

Remark 3. Just like Lévy subordinators, a gamma subordinator is a limiting compound Poisson process.

First, observe that

$$\gamma \log(1+\lambda\theta) = \int_0^{+\infty} (1-\mathrm{e}^{-\lambda x})\rho_\chi(x)\,\mathrm{d}x \qquad \text{with} \quad \rho_\chi(x) = \frac{\gamma}{x}\mathrm{e}^{-x/\theta}.$$

Note that the exponent measure also concentrates around zero, although to a smaller extent; besides it shows exponential decay at infinity.

Considering the compound Poisson process $X^\varepsilon(t)$, $t \geqslant 0$ defined by an exponential interarrival time with mean θ^ε and with IID positive increments, say χ^ε, with common probability density given this time by

$$f_{\chi^\varepsilon}(x) = \theta^\varepsilon \cdot \frac{\gamma}{x}\mathrm{e}^{-x/\theta} \cdot \mathbf{1}(x > \varepsilon) \tag{47}$$

the ε-limiting construction of $X(t)$ suggested in remark 2 can be applied, using this jump distribution instead of the one in (40).

Similarly, the inverse-gamma process $Z(t)$ with Laplace functional

$$\hat{\Phi}_Z(p,\lambda) = \frac{1}{p}\left(1-\lambda\frac{1}{\lambda+\gamma\log(1+\theta p)}\right) = \frac{1}{p(1+\lambda/[\gamma\log(1+\theta p)])} \tag{48}$$

is therefore a limiting renewal process $Z^\varepsilon(t)$ as $\varepsilon \downarrow 0$. Here $Z^\varepsilon(t)$ is a coarse renewal process with space–time characteristics

$$\hat{f}_{\zeta^\varepsilon}(\lambda) = \hat{f}_{\zeta_0^\varepsilon}(\lambda) = \frac{1}{1+\lambda\theta^\varepsilon} \qquad \text{and} \qquad \hat{f}_{\tau^\varepsilon}(p) = \hat{f}_{\chi^\varepsilon}(p) \tag{49}$$

with $\hat{f}_{\chi^\varepsilon}(p) = \langle \mathrm{e}^{-p\chi^\varepsilon}\rangle$ the Laplace transform of the truncated density function f_{χ^ε} defined in (47).

Equation (48) characterizes an inverse-gamma subordinator. Equations (47) and (49) characterize its coarse version $Z^\varepsilon(t)$, $t \geqslant 0$.

Example 4. Composing subordinators. Finally, we supply a last example which shall prove useful in what follows. The idea is to compose subordinators (to obtain a subordinator).

Consider two independent subordinators, say $X(t)$, $Z_1(t)$, $t \geqslant 0$. Assume that $X(t)$ is a Lévy subordinator with parameter $\alpha \in (0,1)$ (see (37) with α replacing δ) and that $Z_1(t)$ is an inverse-Lévy subordinator with parameter δ (given by (43)).

Next, consider the compound subordinator

$$Z(t) := X(Z_1(t)).$$

We have, from Bayes' formula and from (37) and (43)

$$\langle \mathrm{e}^{-\lambda Z(t)}\rangle = \int_0^{+\infty} \langle \mathrm{e}^{-\lambda X(s)}\rangle f_{Z_1(t)}(s)\,\mathrm{d}s = \langle \mathrm{e}^{-\lambda^\delta Z_1(t)}\rangle = \phi_\delta(\lambda^\alpha t^\delta). \tag{50}$$

From (41) and (43), the Laplace functional of $Z(t)$ is thus

$$\hat{\Phi}_Z(p,\lambda) = \frac{1}{p(1+\lambda^\alpha p^{-\delta})}. \tag{51}$$

Hence, from the observations made about Lévy and inverse-Lévy subordinators in example 2, it is the ε-limit Laplace functional of a compound renewal process $Z^\varepsilon(t)$ with local space–time characteristics

$$\hat{f}_{\zeta_0^\varepsilon}(\lambda) = \hat{f}_{\zeta^\varepsilon}(\lambda) = \frac{1}{1+\theta^\varepsilon\lambda^\alpha} \qquad \text{and} \qquad \hat{f}_{\tau^\varepsilon}(p) = \hat{f}_{\chi^\varepsilon}(p).$$

Here $\hat{f}_{\chi^\varepsilon}(p) = \langle \mathrm{e}^{-p\chi^\varepsilon}\rangle$ is the Laplace transform of f_{χ^ε} defined as in the Lévy model (40).

Note that the jump height satisfies $\zeta^\varepsilon \stackrel{d}{=} [\theta^\varepsilon]^{1/\alpha}\zeta$, where ζ is a Mittag–Leffler variable, such that $\Pr(\zeta > x) = \phi_\alpha(x^\alpha)$. Indeed, the Laplace transform $\hat{f}_\zeta(\lambda) := \langle e^{-\lambda\zeta}\rangle$ of such Mittag–Leffler variables are given by the formula [10]

$$\hat{f}_\zeta(\lambda) = \frac{1}{1+\lambda^\alpha}.$$

From (38) and (45), we then have

$$Z(t) \stackrel{d}{=} t^{\delta/\alpha} L_{\delta,\alpha} \qquad t \geqslant 0 \tag{52}$$

where

$$L_{\delta,\alpha} := L_\alpha/(L_\delta)^{\delta/\alpha} = (I_\delta/I_\alpha)^{1/\alpha} \tag{53}$$

with (L_α, L_δ), (respectively, (I_α, I_δ)) two independent Lévy (respectively, inverse-Lévy) variables with respective parameters α and δ.

We shall call the variable $L_{\delta,\alpha}$ a *mixed* inverse-Lévy variable, for obvious reasons. (see [17] for some use of the process (52) in the context of fractional Lévy motions). For such compound subordinators, we observe a competition between the very large jumps which occur in the α–Lévy subordinator and the very long stages of the inverse-δ–Lévy subordinator.

4. Linnik and inverse-Linnik subordinators

As we wish to pay special attention to Linnik processes, this example deserves its own section.

Consider two independent subordinators with stationary independent increments, say $X_1(t)$, $X_2(t)$, $t \geqslant 0$. Assume $X_1(t)$ is a Lévy subordinator and that $X_2(t)$ a gamma subordinator. Next, consider the compound subordinator

$$X(t) := X_1(X_2(t)).$$

We have, from Bayes' formula and from (37) and (46)

$$\langle e^{-\lambda X(t)}\rangle = \int_0^{+\infty} \langle e^{-\lambda X_1(s)}\rangle f_{X_2(t)}(s)\,ds = \langle e^{-\lambda^\delta X_2(t)}\rangle = (1+\theta\lambda^\delta)^{-\gamma t}.$$

If $\theta = \frac{1}{\gamma}$, $X(t), t \geqslant 0$ is said to be a Linnik subordinator. Hence, for such processes, with $\gamma > 0$ and $\delta \in (0, 1)$

$$\langle e^{-\lambda X(t)}\rangle = \langle e^{-\lambda X^*(t)}\rangle = (1+\lambda^\delta/\gamma)^{-\gamma t} = e^{-\gamma t \log(1+\lambda^\delta/\gamma)}. \tag{54}$$

Processes with such distributions are known to be with stationary independent increments, as a result of the ID character of the Linnik distribution defined by its Laplace–Stieltjes–Cole–Cole transform [35]

$$\langle e^{-\lambda X(1)}\rangle = (1+\lambda^\delta/\gamma)^{-\gamma}. \tag{55}$$

In fact, Linnik distributions belong to the class of SD distributions (see below) which is a proper subclass of ID ones.

These distributions were first introduced in [24, p 63] and then developed in the present form by [7, 8]. Recent additional work on this topic may be found in [2–4], in a static context.

Note that as $\gamma = 1$, $X(t)$, $t \geqslant 0$ is a Mittag–Leffler process [28] and that as $\gamma \uparrow +\infty$, $X(t)$, $t \geqslant 0$, coincides with a Lévy-stable subordinator. This class of processes is thus quite general and fundamental in many respects. Furthermore, for large times, the median value $m_{X(t)}$ of $X(t)$, defined by $\Pr(X(t) \leqslant m_{X(t)}) = \frac{1}{2}$ is easily shown, from (54), to grow like $t^{1/\delta}$: this process also drifts to infinity much faster than clock time t.

Remark 4. Just like Lévy (and gamma) subordinators, a Linnik subordinator is a limiting compound Poisson process. From the ID character of the Linnik distribution and from the fact that this compound process is a subordinator

$$\gamma \log(1 + \lambda^\delta/\gamma) = \int_0^{+\infty} (1 - e^{-\lambda x}) \rho_\chi(x)\, dx \tag{56}$$

for some positive Radon measure $\rho_\chi(x)\, dx$. The function $\rho_\chi(x)$ may next be identified to be

$$\rho_\chi(x) = \frac{\delta\gamma}{x} \phi_\delta(\gamma x^\delta)$$

in terms of the Mittag–Leffler function ϕ_δ defined in (44). Indeed, upon deriving expression (56) with respect to λ, we get

$$\int_0^{+\infty} e^{-\lambda x} \phi_\delta(\gamma x^\delta)\, dx = \frac{1}{\lambda(1 + \gamma\lambda^{-\delta})}$$

which is consistent with (41) and (43). Note from this expression that $\phi_\delta(\gamma x^\delta) \sim x^{-\delta}$ for large x, so that $\rho_\chi(x) \sim x^{-(\delta+1)}$ is again heavy-tailed with tail index $\delta \in (0, 1)$: very large jumps in the Lévy subordinator $X_1(t)$ are still visible in the compound Linnik subordinator $X(t)$. In the vicinity of $x = 0$, the exponent measure behaves like $\rho_\chi(x) \sim 1/x$, just like the one of the gamma subordinator.

Considering the compound Poisson process $X^\varepsilon(t)$, $t \geqslant 0$ defined by an exponential interarrival time with mean θ^ε and with IID positive increments, say χ^ε, with common truncated heavy-tailed density given this time by

$$f_{\chi^\varepsilon}(x) = \theta^\varepsilon \cdot \frac{\delta\gamma}{x} \phi_\delta(\gamma x^\delta) \cdot \mathbf{1}(x > \varepsilon) \tag{57}$$

the ε-limiting construction of $X(t)$ suggested in remark 2 applies, using distribution (57) instead of the one (40) used in the Lévy case.

Turning now to the inverse-Linnik subordinator, it is characterized by its Laplace functional

$$\hat{\Phi}_Z(p, \lambda) = \frac{1}{p}\left(1 - \lambda \frac{1}{\lambda + \gamma \log(1 + p^\delta/\gamma)}\right) = \frac{1}{p(1 + \lambda/[\gamma \log(1 + p^\delta/\gamma)])}. \tag{58}$$

Again, from (11), it is the ε-limit of a process $Z^\varepsilon(t)$ whose Laplace functional is the one of a compound renewal process with space–time characteristics

$$\hat{f}_{\zeta_0^\varepsilon}(\lambda) = \hat{f}_{\zeta^\varepsilon}(\lambda) = \frac{1}{1 + \lambda\theta^\varepsilon} \qquad \text{and} \qquad \hat{f}_{\tau^\varepsilon}(p) = \hat{f}_{\chi^\varepsilon}(p). \tag{59}$$

Here, $\hat{f}_{\chi^\varepsilon}(p)$ the Laplace transform of $f_{\chi^\varepsilon}(x)$ given in (57) in terms of the Mittag–Leffler function. The holding times between consecutive exponential jumps are long (events are rare).

4.1. Discrete inverse Cox–Linnik processes

We now come to a discrete version of the inverse-Linnik process just defined.

4.1.1. Discrete inverses of compound Poisson processes. First, let $X(t)$ be a compound Poisson subordinator with positive stationary independent increments. Let $Z(t)$ be its inverse subordinator defined in example 1. It is characterized by its Laplace functional (11) with $(\hat{f}_{\zeta_0}(\lambda) = \hat{f}_\zeta(\lambda), \hat{f}_\tau(p))$ given by the previous formulae (36). Let $P(t)$, $t \geqslant 0$ be a Poisson process with intensity $\frac{1}{a} > 0$, independent of $Z(t)$. We call the integer-valued process

$$\mathfrak{N}(t) := P(Z(t)) \qquad t \geqslant 0 \tag{60}$$

the *discrete* inverse process of $X(.)$. Thus, a discrete inverse process, also called a Cox process, is a Poisson process sampled in independent inverse time $Z(t)$.

For such processes, we have the following proposition.

Proposition 3. *A discrete inverse process is a counting compound renewal process. The time separating consecutive increments has the Laplace–Stieltjes transform $\hat{f}_{\tau}(p)$, whereas the discrete-valued amplitude of the jumps has the Laplace–Stieltjes transform*

$$\hat{f}_{\eta}(\lambda) := \langle e^{-\lambda\eta} \rangle = \hat{f}_{\zeta}\left(\frac{1-e^{-\lambda}}{a}\right). \tag{61}$$

Proof. From Bayes' formula, with $f_{Z(t)}(s)$ the density of $Z(t)$ at $s > 0$, (60) yields

$$\langle e^{-\lambda\mathfrak{N}(t)} \rangle = \int_0^{+\infty} e^{-\frac{s}{a}(1-e^{-\lambda})} f_{Z(t)}(s)\, ds = \langle e^{-\frac{1-e^{-\lambda}}{a} Z(t)} \rangle. \tag{62}$$

Thus, taking the Laplace transform in time, and with as usual

$$\hat{\Phi}_{\mathfrak{N}}(p,\lambda) := \int_0^{+\infty} e^{-pt} \langle e^{-\lambda\mathfrak{N}(t)} \rangle\, dt \tag{63}$$

we get

$$\hat{\Phi}_{\mathfrak{N}}(p,\lambda) = \hat{\Phi}_Z\left(p, \frac{1-e^{-\lambda}}{a}\right). \tag{64}$$

From (11), with $(\hat{f}_{\zeta_0}(\lambda), \hat{f}_{\zeta}(\lambda), \hat{f}_{\tau}(p))$ the space–time characteristics of $Z(.)$, we get

$$\hat{\Phi}_{\mathfrak{N}}(p,\lambda) = \frac{(1-\hat{f}_{\tau}(p))\hat{f}_{\eta_0}(\lambda)}{p(1-\hat{f}_{\tau}(p)\hat{f}_{\eta}(\lambda))} \tag{65}$$

with $\hat{f}_{\eta_0}(\lambda) = \hat{f}_{\eta}(\lambda) = \hat{f}_{\zeta}(\frac{1-e^{-\lambda}}{a})$. □

Remark 5. The process $\mathfrak{N}(t)$ may thus be obtained in the following way:

$$\mathfrak{N}(t) = \sum_{m=0}^{N(t)} \boldsymbol{\eta}_m$$

where $N(t)$ is the elementary counting renewal process with inter-arrival times distributed according to $\hat{f}_{\tau}(p)$ and with $(\boldsymbol{\eta}_m, m \geqslant 0)$ an IID sequence with common Laplace transform $\hat{f}_{\eta}(\lambda) = \hat{f}_{\zeta}(\frac{1-e^{-\lambda}}{a})$ defined in terms of $\hat{f}_{\zeta}(\lambda)$. In this interpretation, at random times T_n, $n \geqslant 1$, a random number of events drawn from $\boldsymbol{\eta}$ simultaneously occur.

4.1.2. Discrete inverse Cox–Linnik processes. We wish here to define a discrete inverse-Cox–Linnik process $\mathfrak{N}(.)$. To do this, we shall first use the coarse process $Z^{\varepsilon}(.)$ (defined by (57) and (59)), the inverse of the coarse Linnik process $X^{\varepsilon}(.)$ in the compound Poisson class. The discrete inverse, say $\mathfrak{N}^{\varepsilon}(.)$ of the coarse Linnik process $X^{\varepsilon}(.)$ may then be derived from the previous section. Finally, the full discrete inverse-Cox–Linnik process $\mathfrak{N}(.)$ will be defined as: $\mathfrak{N}(.) = \lim_{\varepsilon\downarrow 0} \mathfrak{N}^{\varepsilon}(.)$.

At fixed $\varepsilon > 0$, the time separating consecutive increments of $\mathfrak{N}^{\varepsilon}(.)$ has the Laplace–Stieltjes transform

$$\hat{f}_{\tau^{\varepsilon}}(p) = \hat{f}_{\chi^{\varepsilon}}(p) \tag{66}$$

with $\hat{f}_{\chi^{\varepsilon}}(p)$ the Laplace transform of $f_{\chi^{\varepsilon}}(x)$ given in (57) in terms of the Mittag–Leffler function.

The discrete-valued amplitude of the jumps has the geometric Laplace–Stieltjes transform

$$\hat{f}_{\eta^{\varepsilon}}(\lambda) = \hat{f}_{\zeta^{\varepsilon}}\left(\frac{1-\mathrm{e}^{-\lambda}}{a}\right) \qquad \text{with} \quad \hat{f}_{\zeta^{\varepsilon}}(\lambda) = \frac{1}{1+\lambda\theta^{\varepsilon}}. \tag{67}$$

In other words, $\boldsymbol{\eta}^{\varepsilon}$ is geometrically distributed and with $p_{\varepsilon} := \frac{\theta^{\varepsilon}}{a+\theta^{\varepsilon}}$

$$\Pr(\boldsymbol{\eta}^{\varepsilon} = n) = (1-p_{\varepsilon})p_{\varepsilon}^{n} \qquad n \geqslant 0. \tag{68}$$

In addition, $\hat{f}_{\eta_0}^{\varepsilon}(\lambda) = \hat{f}_{\eta}^{\varepsilon}(\lambda)$ and the Laplace functional of the limiting process $\mathfrak{N}(.)$ is

$$\hat{\Phi}_{\mathfrak{N}}(p,\lambda) = \frac{1}{p(1+(1-\mathrm{e}^{-\lambda})/[a\gamma\log(1+p^{\delta}/\gamma)])}. \tag{69}$$

4.2. Self-decomposability

We supply here some notions on the self-decomposability property advocated above of Linnik distributions.

A positive random variable X is said to be SD if it can be additively (self-) decomposed according to

$$X \stackrel{d}{=} c \cdot X_1 + R_0 \tag{70}$$

where $c > 0$ is a scale parameter. Also, X and X_1 share the same distribution and X_1 is independent of the remaining (positive) random variable R_0.

Then, we have

Proposition 4. *A positive random variable X is SD if, with $\hat{f}_X(\lambda) := \langle \mathrm{e}^{-\lambda X}\rangle$ and $c > 0$, there is a (probability) Laplace–Stieltjes transform $\hat{f}_c(\lambda)$ such that*

$$\hat{f}_X(\lambda) = \hat{f}_X(c\lambda)\cdot\hat{f}_c(\lambda). \tag{71}$$

If X is SD, with $R_m \stackrel{d}{=} c^m R_0$, $m \geqslant 0$ independent random variables obtained from R_0 through the scaling

$$\sum_{m=0}^{n} R_m \underset{n\uparrow\infty}{\overset{d}{\rightarrow}} X. \tag{72}$$

Proof. Observing the following convergence in law to zero:

$$c^n X \underset{n\uparrow+\infty}{\overset{d}{\rightarrow}} 0$$

we get the result upon iterating the above decomposition of X. Note that

$$\hat{f}_m(\lambda) := \langle \mathrm{e}^{-\lambda R_m}\rangle = \hat{f}_c(c^m\lambda) = \frac{\hat{f}_X(c^m\lambda)}{\hat{f}_X(c^{m+1}\lambda)}.$$

□

Thus, SD variables derive their importance from the fact that they are limit distributions for sums of independent random variables, through scaling iteratively. They are known [25] to be a sub-class of ID variables X for which

$$-\log\hat{f}_X(\lambda)$$

has a completely monotone λ-derivative [10].

Let now N be an integer-valued random variable. There exists a discrete version of the notion of self-decomposability [33]. Some accounts on the sub-class of discrete stable random variables may also be found here and in [5].

The probability generating function (PGF) $\varphi(u) := \langle u^N \rangle$ is the one of a discrete SD variable N if for any $p \in (0, 1)$, there is a PGF $\varphi_p(u)$ such that

$$\varphi(u) = \varphi(1 - p(1 - u)) \cdot \varphi_p(u). \tag{73}$$

This is the standard (discrete) version of self-decomposability of probability distributions on the integers, through a functional equation. We then have the obvious characterization property given in the following proposition.

Proposition 5. *It follows from the definition of SD distributions that if $\varphi(u)$ is the PGF of the random variable N, then N can be additively decomposed as*

$$N \overset{d}{=} p \circ N_1 + R_0 \tag{74}$$

where the p-thinned random variable $p \circ N$, for $p \in (0, 1]$, is defined to be

$$p \circ N := \sum_{m=1}^{N} B_m \tag{75}$$

with $(B_m, m \geqslant 1)$ IID Bernoulli variables in such a way that $P(B_1 = 1) = p$, independent of N. Also, N and N_1 have the same distribution and $p \circ N_1$ is independent of the remaining random variable R_0.

Observing that for any two real numbers p_1 and p_2 of $(0, 1]$, $p_1 \circ (p_2 \circ N) = (p_1 \cdot p_2) \circ N$, and that the following convergence in law to zero holds:

$$p^n \circ N \underset{n\uparrow+\infty}{\overset{d}{\to}} 0$$

we obtain, iterating the above decomposition

$$\sum_{m=0}^{n} R_m \underset{n\uparrow+\infty}{\overset{d}{\to}} N$$

where $R_m \overset{L}{\equiv} p^m \circ R_0$, $m \geqslant 0$, are independent random variables with PGF

$$\varphi_{R_m}(u) = \varphi_{R_0}(1 - p^m(1 - u)).$$

Here p^m are scaling parameters. Discrete SD random variables are thus also obtained as limit distributions for sums of independent discrete random variables. It can be shown that the SD subclass of ID distributions only yields unimodal distributions, with mode possibly at the origin.

Finally, the connection between positive and discrete self-decomposability relies on the following proposition.

Proposition 6. *Let $P(t)$, $t \geqslant 0$ be a standard Poisson process. If X is a positive SD distribution, the discrete variable N defined by*

$$P(X) = N \tag{76}$$

is discrete SD.

Proof. This goes through the observation employed in the design of discrete processes from continuous ones (section 4.1) that the Laplace–Stieltjes transform $\hat{f}_N(\lambda) := \varphi(\mathrm{e}^{-\lambda})$, $\lambda \geqslant 0$, of N is related to the Laplace–Stieltjes transform $\hat{f}_X(\lambda)$ of X by

$$\hat{f}_N(\lambda) = \hat{f}_X(1 - \mathrm{e}^{-\lambda}).$$

□

5. Asymptotics of the cumulative partial sums of a Linnik sequence

Let $(\boldsymbol{\xi}_n, n \geqslant 0)$ be an IID sequence of positive magnitude. In this section, we are concerned with the large-time behaviour of the cumulated variable $\mathfrak{X}(t)$, where

$$\mathfrak{X}(t) = \sum_{m=0}^{\mathfrak{N}(t)} \boldsymbol{\xi}_m \tag{77}$$

with $\mathfrak{N}(t)$ a discrete Linnik process, defined by (69).

For such processes, from (21) the Laplace functional is given by

$$\hat{\Phi}_{\mathfrak{X}}(p, \lambda) = \hat{\Phi}_{\mathfrak{N}}(p, -\log \hat{f}_{\boldsymbol{\xi}}(\lambda)) \tag{78}$$

with $\hat{\Phi}_{\mathfrak{N}}(p, \lambda)$ given by (69).

It happens that the asymptotics of this variable is strongly dependent on the finiteness or not of the mean value $\mu := \langle \boldsymbol{\xi} \rangle$ for the amplitude. We shall therefore distinguish these two cases.

First we consider the case $\mu < +\infty$.

Proposition 7. *If $\mu < +\infty$, we have (see also [31, p 27]) the convergence in distribution to an inverse-Lévy variable*

$$\frac{a}{t^\delta} \frac{\mathfrak{X}(t)}{\mu} \underset{t\uparrow+\infty}{\overset{d}{\to}} I_\delta. \tag{79}$$

Proof. From the definition (77), the Laplace–Stieltjes functional of $\mathfrak{X}(t)$ results from the one of $\mathfrak{N}(t)$ and

$$\hat{\Phi}_{\mathfrak{X}}(p, \lambda) = \hat{\Phi}_{\mathfrak{N}}(p, \lambda = -\log \hat{f}_{\boldsymbol{\xi}}(\lambda)). \tag{80}$$

From (69), it is

$$\hat{\Phi}_{\mathfrak{X}}(p, \lambda) = \frac{1}{p(1 + (1 - \hat{f}_{\boldsymbol{\xi}}(\lambda))/[a\gamma \log(1 + p^\delta/\gamma)])}. \tag{81}$$

Thus, under the hypothesis $\mu < +\infty$, we have $\hat{f}_{\boldsymbol{\xi}}(\lambda) \underset{\lambda\downarrow 0}{\sim} 1 - \mu\lambda$, and

$$\hat{\Phi}_{\mathfrak{X}}(p, \lambda) \underset{(p,\lambda)\downarrow\mathbf{0}}{\sim} \frac{1}{p(1 + \frac{\mu}{a}\lambda p^{-\delta})} \tag{82}$$

which from (41) and (45) is the Laplace functional of the variable $\frac{\mu}{a} t^\delta I_\delta$. □

From (79), as $\delta < 1$, the cumulated variable $\mathfrak{X}(t)$ grows *slower* than time t, as a result of the long stages in the inverse-Linnik process: the time between consecutive events may be long and this property slows the speed of growth of $\mathfrak{X}(t)$.

Remark 6. This result actually contains two additional obvious results as particular cases. They concern respectively the number of peaks $\mathfrak{N}(t)$ in a Linnik sequence, for which

$$\frac{a}{t^\delta} \mathfrak{N}(t) \underset{t\uparrow+\infty}{\overset{d}{\to}} I_\delta \tag{83}$$

and the number of peaks over the threshold $x > 0$ (POT), say $\overline{\mathfrak{N}}_x(t) := \sum_{m=1}^{\mathfrak{N}(t)} \mathbf{1}(\boldsymbol{\xi}_m > x)$, for which the following is easily shown to hold:

$$\frac{a}{t^\delta} \frac{\overline{\mathfrak{N}}_x(t)}{\Pr(\boldsymbol{\xi} > x)} \underset{t\uparrow+\infty}{\overset{d}{\to}} I_\delta. \tag{84}$$

This variable is of interest for the computation of *order statistics* $\boldsymbol{\xi}_{1:n} \leqslant \cdots \leqslant \boldsymbol{\xi}_{n:n}$, observing that, with $k \geqslant 1$ integer, the events '$\mathfrak{N}_x(t) > k$' and '$\boldsymbol{\xi}_{k:\mathfrak{N}(t)} \leqslant x$' coincide. Here $\mathfrak{N}_x(t) := \sum_{m=1}^{\mathfrak{N}(t)} \mathbf{1}(\boldsymbol{\xi}_m \leqslant x) = \mathfrak{N}(t) - \overline{\mathfrak{N}}_x(t)$. If the number of peaks under the threshold is larger than k, then the amplitude of the kth ordered peak cannot exceed this threshold.

Next we consider the case $\mu = +\infty$.

We now come to the situation where the expected individual magnitude has infinite value as a result of heavy-tailedness of $\boldsymbol{\xi}$

$$\hat{f}_{\boldsymbol{\xi}}(\lambda) \underset{\lambda \downarrow 0}{\sim} 1 - c_\alpha \lambda^\alpha \qquad \text{with} \quad c_\alpha > 0 \qquad \text{and} \qquad \alpha \in (0, 1). \tag{85}$$

In this case the following proposition holds.

Proposition 8. *If $\mu = +\infty$, in the previous sense, the following convergence in distribution holds (see also [20, p 27])*

$$\frac{1}{\kappa_\alpha \cdot t^{\delta/\alpha}} \mathfrak{X}(t) \underset{t \uparrow +\infty}{\overset{d}{\to}} L_{\delta,\alpha}. \tag{86}$$

Here the random variable $L_{\delta,\alpha}$ is the mixed inverse-Lévy variable which appears in (52). The constant κ_α which appears may be expressed as

$$\kappa_\alpha = \left[\frac{c_\alpha}{a}\right]^{1/\alpha} \tag{87}$$

in terms of the constants already encountered.

Proof. Under our hypotheses (85), it results from (81), that

$$\hat{\Phi}_{\mathfrak{X}}(p, \lambda) \underset{(\lambda,p) \downarrow \mathbf{0}}{\sim} \frac{1}{p(1 + \frac{c_\alpha}{a} \lambda^\alpha p^{-\delta})}. \tag{88}$$

Now, from (50)–(52), this is the Laplace functional of the random variable $L_{\delta,\alpha}(t) = \kappa_\alpha t^{\delta/\alpha} L_{\delta,\alpha}$, (see also [17]), proving the result (86). □

From this result, we may now observe a competition between the heavy-tailed character of the magnitude $\boldsymbol{\xi}$ which forces the cumulated variable $\mathfrak{X}(t)$ to grow faster than time t and the heavy-tailed character of the inter-arrival times between consecutive events which slows its growth. A critical situation is when $\delta = \alpha$, in which case, $\mathfrak{X}(t)$ grows linearly with time t and the two effects exactly compensate. If this is the case, (86) reduces to

$$\frac{1}{\kappa_\alpha \cdot t} \mathfrak{X}(t) \underset{t \uparrow +\infty}{\overset{d}{\to}} L_{\alpha,\alpha} \tag{89}$$

where $L_{\alpha,\alpha}$ is the ratio of two independent Lévy variables with the *same* parameter α.

6. Time to failure for Linnik sequences

We finally address the 'time to failure' problem for Linnik sequences announced in the introduction of this paper. Indeed, in some instances, the cumulative magnitude process is irrelevant; rather, an 'extremal' process is required. For example, if $(\boldsymbol{\xi}_n, n \geqslant 0)$ are the tides amplitudes' sequence which a dyke is to contain, it is of concrete interest to have some insight into the waiting time before some tides' height will exceed the dam's height itself, provoking irreversible damage. This approach may then be used to design the dam's height so as to guarantee, with a good confidence interval, that this failure time will be long enough.

6.1. Time to failure as an extreme values problem

Let $(\boldsymbol{\xi}_n, n \geqslant 0)$ be an IID sequence of random positive magnitude. A sequence $(\boldsymbol{\xi}_0, \boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{\mathfrak{N}(t)})$ where $\mathfrak{N}(t)$ is a discrete Linnik process will be called a Linnik *sequence* of magnitude events.

For such sequences, an obvious question of interest is: *how long should one wait before the first event of magnitude greater than a given threshold is observed*?

To that purpose, we shall compute the 'time to failure' in a renewal Linnik sequence, which is defined as

$$T(x) := \inf(t > 0 : \boldsymbol{\xi}_{\mathfrak{N}(t)} > x) \tag{90}$$

i.e. which is the first time at which some magnitude exceeds the level $x > 0$. Our result is given in the following proposition.

Proposition 9. *Let $\mathfrak{N}(t)$ be a discrete inverse-Linnik process. Let $(\boldsymbol{\xi}_n, n \geqslant 0)$ be a sequence of IID random variables with common probability distribution* $\Pr(\boldsymbol{\xi} \leqslant x)$. *Then the Laplace transform of $T(x)$ is*

$$\langle \mathrm{e}^{-pT(x)} \rangle = \frac{1}{1 + [a\gamma \log(1 + p^{\delta}/\gamma)]/\Pr(\boldsymbol{\xi} > x)}$$

which is explicit.

Proof. This problem is one of extreme value theory [9]. To see this, set $X(t) = \boldsymbol{\xi}_{\mathfrak{N}(t)}, t \geqslant 0$. From (90) and from the generalities on inverse processes (section 3.1), the 'time to failure' process $T(x), x \geqslant 0$ is the inverse of the extremal process $X^*(t) := \max_{s \leqslant t} X(s)$, which exhibits non-decreasing sample paths. Here, this extremal process is

$$X^*(t) = \max(\boldsymbol{\xi}_0, \boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{\mathfrak{N}(t)}).$$

Therefore, we shall first express the distribution function for the variable $\max(\boldsymbol{\xi}_0, \boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{\mathfrak{N}(t)})$, where $(\boldsymbol{\xi}_n, n \geqslant 0)$ is the IID random sequence of amplitudes in a compound renewal process. We have, for positive x

$$\begin{aligned} \Pr(\max(\boldsymbol{\xi}_0, \ldots, \boldsymbol{\xi}_{\mathfrak{N}(t)}) \leqslant x) &= \sum_{n \geqslant 0} \Pr(\max(\boldsymbol{\xi}_0, \ldots, \boldsymbol{\xi}_n) \leqslant x) \Pr(\mathfrak{N}(t) = n) \\ &= \sum_{n \geqslant 0} [\Pr(\boldsymbol{\xi} \leqslant x)]^n \Pr(\mathfrak{N}(t) = n) = \Phi_{\mathfrak{N}}(t, -\log \Pr(\boldsymbol{\xi} \leqslant x)) \end{aligned} \tag{91}$$

from the definition of function $\Phi_{\mathfrak{N}}(t, \lambda)$.

Now, the event '$\max(\boldsymbol{\xi}_0, \boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{\mathfrak{N}(t)}) \leqslant x$' obviously coincides with the event '$T(x) > t$' so that (91) gives the probability distribution of $T(x)$. In other words, using (64)

$$\int_0^{+\infty} \mathrm{e}^{-pt} \Pr(T(x) > t)\, \mathrm{d}t = \hat{\Phi}_{\mathfrak{N}}(p, -\log \Pr(\boldsymbol{\xi} \leqslant x)) = \hat{\Phi}_Z\left(p, \frac{\Pr(\boldsymbol{\xi} > x)}{a}\right). \tag{92}$$

Here, $\hat{\Phi}_Z(p, \lambda)$ is the Laplace functional of an inverse-Linnik subordinator given by (58). Now

$$\int_0^{+\infty} \mathrm{e}^{-pt} \Pr(T(x) > t)\, \mathrm{d}t = \frac{1}{p}(1 - \langle \mathrm{e}^{-pT(x)} \rangle) \tag{93}$$

so that, from (58)

$$\langle \mathrm{e}^{-pT(x)} \rangle = 1 - p\hat{\Phi}_Z\left(p, \frac{\Pr(\boldsymbol{\xi} > x)}{a}\right)$$

$$= 1 - \frac{1}{(1 + \Pr(\boldsymbol{\xi} > x)/[a\gamma \log(1 + p^\delta/\gamma)])}$$
$$= \frac{1}{1 + [a\gamma \log(1 + p^\delta/\gamma)]/\Pr(\boldsymbol{\xi} > x)}. \tag{94}$$

We thus proved the assertion. □

Remark 7. If $x = m_{\boldsymbol{\xi}}$ is the median value for $\boldsymbol{\xi}$ defined by $\Pr(\boldsymbol{\xi} \leqslant m_{\boldsymbol{\xi}}) = \frac{1}{2}$ then

$$\langle \mathrm{e}^{-pT(m_{\boldsymbol{\xi}})} \rangle = \frac{1}{1 + 2[a\gamma \log(1 + p^\delta/\gamma)]}. \tag{95}$$

6.2. Limit law for large threshold values

In some applications, one is interested in the first time by which some amplitude exceeds the level x, where x is itself assumed to be 'large'. Large here means that $\Pr(\boldsymbol{\xi} > x)$ is close to zero, so that crossing the threshold x is very unlikely. Let us therefore discuss the asymptotics of $T(x)$ for large $x > 0$.

Proposition 10. *The following convergence in distribution holds:*

$$[\Pr(\boldsymbol{\xi} > x)/a]^{1/\delta} T(x) \underset{x\uparrow+\infty}{\overset{d}{\to}} M_\delta \tag{96}$$

where

$$\langle \mathrm{e}^{-qM_\delta} \rangle = \frac{1}{1 + q^\delta} \qquad q \geqslant 0. \tag{97}$$

Thus, the limit variable M_δ is a Mittag–Leffler variable

$$\Pr(M_\delta > x) = \phi_\delta(x^\delta) \tag{98}$$

already encountered.

Proof. As $x \uparrow +\infty$, we have $\Pr(\boldsymbol{\xi} > x) \downarrow 0$. Thus,

$$\langle \mathrm{e}^{-q[\Pr(\boldsymbol{\xi}>x)/a]^{1/\delta} T(x)} \rangle$$
$$= \frac{1}{1 + [a\gamma \log(1 + \Pr(\boldsymbol{\xi} > x) q^\delta/(a\gamma))]/\Pr(\boldsymbol{\xi} > x)} \underset{x\uparrow+\infty}{\to} \frac{1}{1 + q^\delta}$$

as required. □

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